

A generalization of Matkowski's fixed point theorem and Istrăţescu's fixed point theorem concerning convex contractions

Radu MICULESCU and Alexandru MIHAIL

Abstract. In this paper we obtain a generalization of Matkowski's fixed point theorem and Istrăţescu's fixed point theorem concerning convex contractions. More precisely, given a complete b -metric space (X, d) , we prove that every continuous function $f : X \rightarrow X$ is a Picard operator, provided that there exist $m \in \mathbb{N}^*$ and a comparison function φ such that $d(f^{[m]}(x), f^{[m]}(y)) \leq \varphi(\max\{d(x, y), d(f(x), f(y)), \dots, d(f^{[m-1]}(x), f^{[m-1]}(y))\})$ for all $x, y \in X$. In addition, we point out that if $m = 1$, the continuity condition on f is not necessary and consequently, taking into account that a metric space is a b -metric space, we obtain a generalization of Matkowski's fixed point theorem. Moreover, we prove that Istrăţescu's fixed point theorem concerning convex contractions is a particular case of our result for $m = 2$. By providing appropriate examples we show that the above mentioned two generalizations are effective.

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1. Introduction

One branch of generalizations of the celebrated Banach-Cacciopoli-Picard contraction principle is based on the replacement of the contractivity condition imposed on the function $f : X \rightarrow X$, where (X, d) is a complete metric space, by a weaker one described by the inequality $d(f(x), f(y)) \leq \varphi(d(x, y))$ for all $x, y \in X$, where φ has certain properties (see [8], [9], [17] and [23]). The result obtained by J. Matkowski in this direction can be stated as follows: *Every φ -contraction $f : X \rightarrow X$, where (X, d) is a complete metric space, is a Picard operator.*

The notion of b -metric space was introduced by I. A. Bakhtin [4] and S. Czerwik (see [11] and [12]) in connection with some problems concerning the convergence of measurable functions with respect to measure. Among the fixed point results in the framework of b -metric spaces obtained in the last period (see, for example, [2], [6], [7], [10], [18], [19], [22], [25], [28] and the references therein, to mention just the most recent ones) we point out a

Matkowski type fixed point result (in the framework of such a space X with the property that the b -metric is a continuous functional on $X \times X$ and for a b -comparison function φ) which is due to M. Păcurar (Berinde) (see [20] and [21]).

Motivated by the fact that in some situations there is no need to use the entire force of metric requirements in the proof of certain fixed point theorems, J. Jachymski, J. Matkowski and T. Świątkowski [16] obtained a generalization of Matkowski's result for the class of semi-metric spaces satisfying the so called JMS condition (see [3]). Note that this class is larger than the one of b -metric spaces. Another proof of this result (which is based on a monotone principle of fixed points -see [26]-) was given by M. Tasković [27]. By defining the notion of τ -distance function in a general topological space (X, τ) , M. Aauri and D. El. Montawokil [1] obtained a generalization of the result due to Jachymski, Matkowski and Świątkowski. A related result (namely an extension of the Matkowski fixed point theorem in the framework of complete and regular semi-metric spaces) is given in [5].

In an attempt to study if there exist contraction-type conditions that do not imply the contraction condition and for which the existence and uniqueness of the fixed point are assured, V. Istrăţescu introduced and studied the convex contraction condition (see [13], [14] and [15]). A continuous function $f : (X, d) \rightarrow (X, d)$, where (X, d) is a complete metric space, is called convex contraction if there exist $a, b \in (0, 1)$ such that $a + b < 1$ and $d(f^{[2]}(x), f^{[2]}(y)) \leq ad(f(x), f(y)) + bd(x, y)$ for all $x, y \in X$. Istrăţescu proved that any convex contraction has a unique fixed point $\alpha \in X$ (and $\lim_{n \rightarrow \infty} f^{[n]}(x) = \alpha$ for every $x \in X$) and provided an example of convex contraction which is not contraction.

In this paper we prove the following result: Every continuous function $f : X \rightarrow X$, where (X, d) is a complete b -metric space, is a Picard operator, provided that there exist $m \in \mathbb{N}^*$ and a comparison function φ such that $d(f^{[m]}(x), f^{[m]}(y)) \leq \varphi(\max\{d(x, y), d(f(x), f(y)), \dots, d(f^{[m-1]}(x), f^{[m-1]}(y))\})$ for all $x, y \in X$. Moreover, we show that if $m = 1$, then the continuity of f is not necessary and therefore we obtain Matkowski's fixed point theorem as a particular case (by taking into account the fact that every metric space is a b -metric space). In addition, we proved that Istrăţescu's fixed point theorem concerning convex contractions is a particular case of our result for $m = 2$. We provide two examples to justify that the above mentioned two generalizations are effective.

2. Preliminaries

In this section we recall some basic facts that will be used in the sequel.

Definition 2.1. *Given a nonempty set X and a real number $s \in [1, \infty)$, a function $d : X \times X \rightarrow [0, \infty)$ is called b -metric if it satisfies the following properties:*

- i) $d(x, y) = 0$ if and only if $x = y$;
- ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- iii) $d(x, y) \leq s(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called b -metric space (with constant s).

Besides the classical spaces $l^p(\mathbb{R})$ and $L^p[0, 1]$, where $p \in (0, 1)$, more examples of b -metric spaces could be found in [6], [11] and [12].

Remark 2.1. *Every metric space is a b -metric space (with constant 1). There exist b -metric spaces which are not metric spaces (see, for example, [10] or [19]).*

Definition 2.2. *A sequence $(x_n)_{n \in \mathbb{N}}$ of elements from a b -metric space (X, d) is called:*

- convergent if there exists $l \in \mathbb{R}$ such that $\lim_{n \rightarrow \infty} d(x_n, l) = 0$;
- Cauchy if $\lim_{m, n \rightarrow \infty} d(x_m, x_n) = 0$, i.e. for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$

such that $d(x_m, x_n) < \varepsilon$ for all $m, n \in \mathbb{N}$, $m, n \geq n_\varepsilon$.

The b -metric space (X, d) is called complete if every Cauchy sequence of elements from (X, d) is convergent.

Remark 2.2. *A b -metric space can be endowed with the topology induced by its convergence.*

For a function $f : X \rightarrow X$ and $n \in \mathbb{N}$, by $f^{[n]}$ we mean the composition of f by itself n times.

Definition 2.3. *A function $\varphi : [0, \infty) \rightarrow [0, \infty)$ is called a comparison function if:*

- i) φ is increasing;
- ii) $\lim_{n \rightarrow \infty} \varphi^{[n]}(r) = 0$ for every $r \in [0, \infty)$.

Remark 2.3. Every comparison function φ has the property that $\varphi(0) = 0$ and $\varphi(r) < r$ for every $r \in (0, \infty)$.

Definition 2.4. Given a b -metric space (X, d) and a comparison function φ , a function $f : X \rightarrow X$ is called φ -contraction if $d(f(x), f(y)) \leq \varphi(d(x, y))$ for all $x, y \in X$.

Definition 2.5. Given a b -metric space (X, d) , a function $f : X \rightarrow X$ is called Picard operator if there exists a unique fixed point α of f and the sequence $(f^{[n]}(x))_{n \in \mathbb{N}}$ is convergent to α for every $x \in X$.

Lemma 2.1. Given a b -metric space (X, d) with constant $s \geq 1$, the inequality $d(x_0, x_p) \leq \sum_{i=1}^p s^i d(x_{i-1}, x_i)$ is valid for every $p \in \mathbb{N}$ and for all $x_0, x_1, \dots, x_p \in X$.

Proof. We have

$$\begin{aligned} d(x_0, x_p) &\leq sd(x_0, x_1) + sd(x_1, x_p) \leq sd(x_0, x_1) + s^2d(x_1, x_2) + s^2d(x_2, x_p) \leq \\ &\leq sd(x_0, x_1) + s^2d(x_1, x_2) + s^3d(x_2, x_3) + \dots + s^{p-1}d(x_{p-2}, x_{p-1}) + s^{p-1}d(x_{p-1}, x_p) \leq \\ &\leq sd(x_0, x_1) + s^2d(x_1, x_2) + \dots + s^{p-1}d(x_{p-2}, x_{p-1}) + s^pd(x_{p-1}, x_p) = \sum_{i=1}^p s^i d(x_{i-1}, x_i), \end{aligned}$$

for every $p \in \mathbb{N}$ and all $x_0, x_1, \dots, x_p \in X$. \square

3. The main result

Theorem 3.1. Every continuous function $f : X \rightarrow X$, where (X, d) is a complete b -metric space, is a Picard operator, provided that there exist $m \in \mathbb{N}^*$ and a comparison function φ such that the following inequality:

$$d(f^{[m]}(x), f^{[m]}(y)) \leq \varphi(\max\{d(x, y), d(f(x), f(y)), \dots, d(f^{[m-1]}(x), f^{[m-1]}(y))\}),$$

is valid for all $x, y \in X$.

Proof. Let us denote the constant of the b -metric space (X, d) by s .

In the sequel, for $x, y \in X$ and $n \in \mathbb{N}$, we adopt the following notations:

- i) $x_n := f^{[n]}(x)$ and $y_n := f^{[n]}(y)$;
- ii) $M_n(x, y) := \max\{d(x_n, y_n), d(x_{n+1}, y_{n+1}), \dots, d(x_{n+m-1}, y_{n+m-1})\}$.

Claim 1. The sequence $(M_n(x, y))_{n \in \mathbb{N}}$ is decreasing.

Justification of Claim 1. Taking into the inequality from hypothesis $x = x_n$ and $y = y_n$ we get that $d(x_{m+n}, y_{m+n}) \leq \varphi(M_n(x, y)) \stackrel{\text{Remark 2.3}}{\leq} M_n(x, y)$, so $M_{n+1}(x, y) \leq M_n(x, y)$ for every $n \in \mathbb{N}$.

Claim 2. $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.

Justification of Claim 2. For every $i \in \{0, 1, \dots, m-1\}$ and $n \in \mathbb{N}$, taking $x = x_{n+i}$ and $y = y_{n+i}$ into the inequality from hypothesis, we get that $d(x_{m+n+i}, y_{m+n+i}) \leq \varphi(M_{n+i}(x, y)) \stackrel{\text{Claim 1}}{\leq} \varphi(M_n(x, y))$, so $M_{m+n}(x, y) \leq \varphi(M_n(x, y))$. Using the mathematical induction method, we obtain that $M_{n+km}(x, y) \leq \varphi^{[k]}(M_n(x, y))$ for all $k, n \in \mathbb{N}$ and since $\lim_{k \rightarrow \infty} \varphi^{[k]}(M_n(x, y)) = 0$ we infer that $\lim_{k \rightarrow \infty} M_{n+km}(x, y) = 0$. Using Claim 1 we deduce that $\lim_{n \rightarrow \infty} M_n(x, y) = 0$ and since $d(x_n, y_n) \leq M_n(x, y)$ for every $n \in \mathbb{N}$, we conclude that $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.

By taking $y = f(x)$, from Claim 2, we obtain that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (1)$$

The above inequality assures us that there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq 1, \quad (2)$$

for every $n \in \mathbb{N}$, $n \geq n_0$.

Claim 3. *The sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy.*

Justification of Claim 3. Let us suppose, by reductio ad absurdum, that $(x_n)_{n \in \mathbb{N}}$ is not Cauchy. Then there exists $\varepsilon_0 > 0$ having the property that for every $k \in \mathbb{N}$ there exist $m_k, n_k \in \mathbb{N}$, $m_k, n_k > k$ such that $d(x_{n_k}, x_{m_k}) \geq \varepsilon_0$. Hence we get two subsequences $(x_{n_k})_{k \geq 1}$ and $(x_{m_k})_{k \geq 1}$ of $(x_n)_{n \in \mathbb{N}}$ satisfying the following properties:

- a) $n_1 \geq n_0$;
 - b) $d(x_{n_k}, x_{m_k}) \geq \varepsilon_0$;
 - c) $n_k < m_k = \min\{n_k + p \mid p \in \mathbb{N} \text{ and } d(x_{n_k}, x_{n_k+p}) \geq \varepsilon_0\}$.
- In the sequel we adopt the following notation: $C := s^3(2^{\frac{s^m-1}{s-1}} + \varepsilon_0 + 1)$.

Let us note that

$$d(x_{n_k}, x_{m_k}) < s(\varepsilon_0 + 1) \leq C, \quad (3)$$

for every $k \in \mathbb{N}$.

Indeed, we have $d(x_{n_k}, x_{m_k}) \leq sd(x_{n_k}, x_{m_k-1}) + sd(x_{m_k-1}, x_{m_k}) \stackrel{c) \& (2)}{\leq} s(\varepsilon_0 + 1)$ for every $k \in \mathbb{N}$.

Moreover we have

$$d(x_{n_k+i}, x_{m_k+i}) \leq C, \quad (4)$$

for every $k \in \mathbb{N}$ and every $i \in \{1, 2, \dots, m-1\}$.

Indeed, we have

$$\begin{aligned} d(x_{n_k+i}, x_{m_k+i}) &\leq s^2(d(x_{n_k+i}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k+i})) \leq \\ &\stackrel{\text{Lemma 2.1}}{\leq} s^2\left(\sum_{l=1}^i s^l d(x_{n_k+l-1}, x_{n_k+l}) + d(x_{n_k}, x_{m_k}) + \sum_{l=1}^i s^l d(x_{m_k+l-1}, x_{m_k+l})\right) \leq \\ &\stackrel{(2)}{\leq} s^2\left(2\sum_{l=1}^i s^l + d(x_{n_k}, x_{m_k})\right) \stackrel{(3)}{\leq} s^2\left(2s\frac{s^i-1}{s-1} + s(\varepsilon_0 + 1)\right) \leq C, \end{aligned}$$

for every $k \in \mathbb{N}$ and every $i \in \{1, 2, \dots, m-1\}$.

Inequalities (3) and (4) guarantee that

$$M_0(x_{n_k}, x_{m_k}) \leq C, \quad (5)$$

for every $k \in \mathbb{N}$.

As φ is a comparison function, $\lim_{n \rightarrow \infty} \varphi^{[n]}(C) = 0$, so there exists $p \in \mathbb{N}$ such that

$$s^2 \varphi^{[p]}(C) < \frac{\varepsilon_0}{3}. \quad (6)$$

Now we choose $\varepsilon_1 > 0$ such that

$$2\varepsilon_1 s^3 \frac{s^{pm} - 1}{s - 1} < \frac{\varepsilon_0}{3}. \quad (7)$$

Moreover, taking into account (1), there exists $n_{\varepsilon_1} \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \varepsilon_1, \quad (8)$$

for every $n \in \mathbb{N}$, $n \geq n_{\varepsilon_1}$.

Then, for $k \in \mathbb{N}$ such that $n_k > n_{\varepsilon_1}$, we have:

$$\stackrel{b)}{\varepsilon_0} \leq d(x_{n_k}, x_{m_k}) \leq s^2 d(x_{n_k}, x_{n_k+pm}) + s^2 d(x_{n_k+pm}, x_{m_k+pm}) + s^2 d(x_{m_k+pm}, x_{m_k}) \leq$$

$$\begin{aligned}
& \stackrel{\text{Lemma 2.1}}{\leq} s^2 \varphi^{[p]}(M_0(x_{n_k}, x_{m_k})) + s^2 \sum_{i=1}^{pm} s^i (d(x_{n_k+i-1}, x_{n_k+i}) + d(x_{m_k+i-1}, x_{m_k+i})) \leq \\
& \stackrel{(8)}{\leq} 2\varepsilon_1 s^2 \sum_{i=1}^{pm} s^i + s^2 \varphi^{[p]}(M_0(x_{n_k}, x_{m_k})) \stackrel{(5)}{\leq} 2\varepsilon_1 s^3 \frac{s^{pm} - 1}{s - 1} + s^2 \varphi^{[p]}(C) \stackrel{(6) \& (7)}{\leq} \frac{2\varepsilon_0}{3}.
\end{aligned}$$

This contradiction closes the justification of the claim.

Since (X, d) is a complete b -metric space, Claim 3 assures us that there exists $\alpha \in X$ such that

$$\lim_{n \rightarrow \infty} x_n = \alpha. \quad (9)$$

As f is continuous, from (9) we infer that $f(\alpha) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} x_{n+1}$, so

$$\lim_{n \rightarrow \infty} x_n = f(\alpha). \quad (10)$$

Relations (9) and (10) imply that $f(\alpha) = \alpha$, i.e. α (which is the limit of the sequence $(f^{[n]}(x))_{n \in \mathbb{N}}$ with x arbitrarily chosen in X) is a fixed point of f .

In addition, α is the unique fixed point of f .

Indeed, if $\beta \in X \setminus \{\alpha\}$ would be a fixed point of f , then we arrive at the following contradiction: $0 < d(\alpha, \beta) = d(f^{[m]}(\alpha), f^{[m]}(\beta)) \leq \varphi(d(\alpha, \beta), d(f(\alpha), f(\beta)), \dots, d(f^{[m]}(\alpha), f^{[m]}(\beta))) = \varphi(d(\alpha, \beta)) \stackrel{\text{Remark 2.3}}{<} d(\alpha, \beta)$. \square

Corollary 3.1 (Matkowski's fixed point theorem in the framework of b -metric spaces). *Every φ -contraction $f : X \rightarrow X$, where (X, d) is a complete b -metric space and φ a comparison function, is a Picard operator.*

Proof. The proof is exactly as the one of the above result, except the justification of (10) which is the following one: because $d(x_{n+1}, f(\alpha)) = d(f(x_n), f(\alpha)) \leq \varphi(d(x_n, \alpha)) \stackrel{\text{Remark 2.3}}{\leq} d(x_n, \alpha)$ for every $n \in \mathbb{N}$, using (9), we come to the conclusion that $\lim_{n \rightarrow \infty} x_n = f(\alpha)$. \square

Example 3.1. The function $f : [0, 1] \rightarrow [0, 1]$ given by $f(x) = \begin{cases} \frac{x}{4}, & x \in [0, \frac{1}{2}) \\ \frac{x}{5}, & x \in [\frac{1}{2}, 1] \end{cases}$ -see [13]- has the property that $d(f^{[2]}(x), f^{[2]}(y)) \leq \varphi(\max\{d(x, y), d(f(x), f(y))\})$ for all $x, y \in [0, 1]$, where φ is the comparison function given by $\varphi(r) = \frac{1}{4}r$ for every $r \in [0, \infty)$, so it satisfies the hypothesis of Theorem 3.1. Since f is

not continuous, it does not satisfy the hypothesis of Matkowski's Theorem. Thus *Theorem 3.1 is an effective generalization of Matkowski's fixed point theorem.*

Corollary 3.2 (Istrătescu's fixed point theorem concerning convex contractions in the framework of b-metric spaces). *Given a complete b-metric space (X, d) , every convex contraction $f : X \rightarrow X$ (i.e. there exist $a, b \in (0, 1)$ such that $a+b < 1$ and $d(f^{[2]}(x), f^{[2]}(y)) \leq ad(f(x), f(y)) + bd(x, y)$ for all $x, y \in X$) is a Picard operator.*

Proof. By considering the comparison function $\varphi : [0, \infty) \rightarrow [0, \infty)$ given by $\varphi(r) = (a + b)r$ for every $r \in [0, \infty)$, we have

$$d(f^{[2]}(x), f^{[2]}(y)) \leq ad(f(x), f(y)) + bd(x, y) \leq$$

$$\leq (a + b) \max\{d(x, y), d(f(x), f(y))\} = \varphi(\max\{d(x, y), d(f(x), f(y))\}),$$

for all $x, y \in X$. Now we just apply Theorem 3.1 for $m = 2$. \square

Example 3.2. The function $f : [0, \frac{1}{2}] \rightarrow [0, \frac{1}{2}]$, given by $f(x) = x - x^2$ for every $x \in [0, \frac{1}{2}]$, is not a convex contraction since if this is not the case, then there exist $a, b \in (0, 1)$ such that $a + b < 1$ and $|f^{[2]}(x) - f^{[2]}(y)| \leq a|f(x) - f(y)| + b|x - y|$ for all $x, y \in [0, \frac{1}{2}]$. Picking $x \in [0, \frac{1}{2}]$, one can easily check that the sequence $(x_n)_{n \in \mathbb{N}}$, given by $x_n = f^{[n]}(x)$ for every $n \in \mathbb{N}$, satisfies the following two properties: a) $x_n \leq (a + b)^{[\frac{n}{2}]}$ for every $n \in \mathbb{N}$; b) $\lim_{n \rightarrow \infty} nx_n = 1$. Consequently $nx_n \leq n(a + b)^{[\frac{n}{2}]}$ for every $n \in \mathbb{N}$ and by passing to limit as n goes to ∞ we obtain the contradiction $1 \leq 0$. Thus f does not satisfy the hypothesis of Istrătescu's fixed point theorem concerning convex contractions.

Considering the comparison function $\varphi : [0, \infty) \rightarrow [0, \infty)$ given by $\varphi(x) = \begin{cases} x - x^2, & x \in [0, \frac{1}{2}] \\ \frac{1}{4}, & x \in (\frac{1}{2}, \infty) \end{cases}$, one can easily check that $|f(x) - f(y)| \leq \varphi(|x - y|)$ for all $x, y \in [0, \frac{1}{2}]$. Consequently $|f^{[2]}(x) - f^{[2]}(y)| \leq \varphi(|f(x) - f(y)|) \leq \varphi(\max\{|x - y|, |f(x) - f(y)|\})$ for all $x, y \in [0, \frac{1}{2}]$. Hence f satisfies the hypothesis of Theorem 3.1 for $m = 2$.

Therefore *Theorem 3.1 (for $m = 2$) is an effective generalization of Istrătescu's fixed point theorem concerning convex contractions.*

Remark 3.1. *Theorem 3.1 gives a partial answer, in the framework of b-metric spaces, to Problem 9.3.1 b) from [24].*

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University of Bucharest
 Faculty of Mathematics and Computer Science
 Str. Academiei 14, 010014 Bucharest, Romania
 E-mail: miculesc@yahoo.com, mihail_alex@yahoo.com